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Critical behaviour for mixed site-bond directed percolation

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Abstract. We study mixed site-bond directed percolation on 2D and 3D lattices by using time-dependent simulations. Our results are compared with rigorous bounds recently obtained by Liggett and by Katori and Tsukahara. The critical fractions p^c_{site} and p^c_{bond} of sites and bonds are extremely well approximated by a relationship reported earlier for isotropic percolation, $(\log p^c_{site}/\log p^{c^*}_{site} + \log p^c_{bond}/\log p^{c^*}_{bond} = 1)$, where $p^{c^*}_{site}$ and $p^{c^*}_{bond}$ are the critical fractions in pure site and bond directed percolation.

The mixed site-bond percolation is a natural extension of the ordinary bond percolation and site percolation models. The model is defined by opening sites of a lattice with probability p_{site} and bonds with probability p_{bond}, clusters are defined as combinations of open sites connected by open bonds. The system is percolating in a region of p_{site} and p_{bond} values in which an infinite cluster exists.

The mixed site-bond percolation is a natural model for physical phenomena in randomly restricted media (accounted for by closed sites) with random iterations (accounted for by open and closed bonds). Introduced by Reynolds *et al* [1], it was used in treating polymer gelation, capillary phenomena and in fracture theory (see [2] and references therein).

Current work is concerned with the directed version of the mixed site-bond percolation model (figure 1), first introduced by Kinzel [3, 4], in which bonds can only be open in a certain direction, physically it can be accounted for by the existence of an external field, restricting interactions allowed in the system (as in the case of transport of particles in porous media, induced by an external electrical field), or it could also be applied to time-dependent phenomena, when movement in a chosen direction corresponds to development in time (see [4–6]).

Both pure bond and pure site directed percolation are known to belong to the same universality class known as DP (directed percolation) universality class. The DP universality class is known to be very robust; with a few exceptions all dynamical particle systems involving extinction–survival transition belong to the DP universality class (see [7, 8] and references therein). The mixed directed side-bond percolation should be expected to belong to the DP universality, although no numerical check to confirm it has ever been conducted.

Recently, Katori and Tsukahara [9] proposed a lower boundary for the phase transition line formed by p_{site} and p_{bond} values, at which the percolation transition takes place in the case of mixed directed percolation in $1+1$ dimensions. On the other hand, exact upper and lower bounds for the phase transition line in $1+1$ dimensions have been recently proposed by Liggett [10].

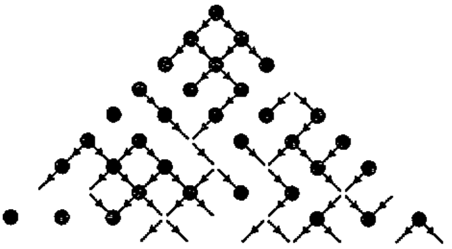

Figure 1. Mixed directed site-bond percolation in $1+1$ dimensions. Percolation is allowed in directions indicated by the arrows. One can clearly see a finite cluster, consisting of nine sites, connected to the origin.

Based on a heuristic argument, Yanuka and Englman [2] proposed an interpolation formula for the phase transition line for ordinary (non-directed) mixed site-bond percolation in any dimension, allowing us to calculate the phase diagram for the mixed percolation from the critical fractions p^{c*}_{site} and p^{c*}_{bond} for pure site and pure bond percolation, corresponding to the endpoints of the diagram. Because of the random spread on data, the Monte Carlo results for the intermediate region reported by the same authors (for a number of lattices in 2D and 3D) do not allow us to see any systematic difference from the phase transition line given by the formula, so that the precision of the critical line given by the formula is unclear.

The principal aim of the current work is the evaluation of the phase transition line for the directed mixed site-bond percolation in $1+1$ and in $2+1$ dimensions using Monte Carlo simulations. For $1+1$ dimensions we confirm the validity of upper and lower bounds suggested by Katori and Tsukahara [9] and by Liggett [10], and, for both $1+1$ and $2+1$ dimensions, show that an interpolation formula, similar to the one suggested by Yanuka and Englman [2] for the ordinary mixed site-bond percolation, is applicable with high precision, although we are able to detect small systematic deviations. The time-dependent simulation approach used in the current work allows, simultaneously with the transition line, the evaluation of dynamical critical exponents for the mixed directed percolation. Our results imply that, not surprisingly, the mixed case belongs to the same universality class to which both pure bond and pure site directed percolation belong.

To decide the critical line, we use a time-dependent simulation [11]. Mixed directed percolation is regarded as a growth process, starting from a single site, the 'origin'. For $1+1$ dimensions, space and time are defined as coordinates of sites on a square lattice perpendicular and parallel to the $(1, 1)$ diagonal, and growth clusters are formed by sites that can be reached from the origin moving through open sites and bonds in the positive time direction (see figure 1). Similarly, for the $2+1$ case we consider growth on a simple cubic lattice with time axis directed along the $(1, 1, 1)$ diagonal. This kind of growth can be realized by updating states of sites of a triangular lattice (see [12] for a discussion of the technique). We calculated the survival probability $P(t)$ as the fraction of realizations containing sites connected to the origin at time t (realizations, surviving at time t); the average number of sites connected to the origin at time t, $n(t)$, averaged over all realizations; and the average square radius $R^2(t)$, averaged over surviving realizations. On the critical line, for large enough t, it is expected that these quantities are governed by power laws

$$P(t) \propto t^{-\delta} \qquad n(t) \propto t^{\eta} \qquad R^2(t) \propto t^{z}. \tag{1}$$

The results of the time-dependent simulation are most conveniently represented as local slopes data (see [11] for a detailed description), for example, for the number of sites connected to the origin we plot $\eta(t) \equiv \ln[n(t)/n(t/m)]/\ln[m]$ against $1/t$, taking $m = 7$. For p_{site} and p_{bond} on the critical line the true η value is given by the intersection of the local slopes curve with ordinata, while in off-critical simulation the local slopes curves

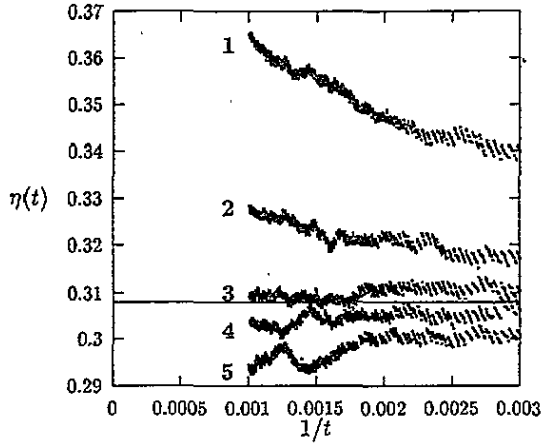

Figure 2. 1+1 dimensions, local slopes for the mean number of sites connected to the origin at different values of p_{site}, $p_{bond} = 1$ for all curves. Line shows the DP value of the exponent η, taken from [7]. (i) $p_{site} = 0.706\,522$ (critical value according to [14]), (ii) $p_{site} = 0.7057$, (iii) $p_{site} = 0.705\,489$ (critical value according to [13]), (iv) $p_{site} = 0.7053$, (v) $p_{site} = 0.7051$.

Table 1. *Critical line in 1 + 1 dimensions, Monte Carlo results.*

p^c_{bond}	p^c_{site}
0.644 701[a]	1
0.7	0.935 85
0.75	0.885 65
0.8	0.841 35
0.85	0.801 90
0.9	0.766 45
0.95	0.734 50
1	0.705 489[a]

[a] Values for ordinary directed percolation are taken from [15, 13].

diverge for small $1/t$. Survival probability P and the mean-square radius R are treated in a similar way. Although the local slopes curves for R are quite insensitive to a change in p_{site} and p_{bond} values and thus less useful in determining the critical line than in the cases of n and P, we used the mean square radius data to check the consistency of our simulation and to estimate the z values.

An example of local slopes curves for time dependent simulation is given in figure 2. The data corresponds to pure site directed percolation, and allows us to support the critical point value provided by Onody and Neves [13] using series technique as opposed to the result obtained by ben-Avraham *et al* [14] using the transfer matrix technique. We only show the local slopes curves for the mean number of sites connected to the origin. Here, and in the rest of the current work, we take 1000 steps and we average over 10^5 configurations. The result strongly implies that the the critical value obtained by Onody and Neves is correct, while the value provided by ben-Avraham *et al* [14] seems to be off the critical point by more than the error margin, given in [14].

The overall results are given in table 1 and in table 2, all p^c_{bond} values given have an uncertainty of ± 0.0002. The uncertainty has been determined by making sure that the local slopes curves for n and P, taken at p_{bond} value, shifted from the one provided in tables 1 and 2 by the value of the error margin, clearly diverge as $1/t$ approaches zero.

Repeating the argument proposed by Yanuka and Englman [2] in the case of mixed

Table 2. Critical line in $2+1$ dimensions, for a simple cubic lattice. Monte Carlo results.

p^c_{bond}	p^c_{site}
0.382 16[a]	1
0.45	0.864 10
0.5	0.787 25
0.6	0.671 15
0.7	0.587 70
0.8	0.524 60
0.9	0.475 10
1	0.435 25[a]

[a] Values for ordinary directed percolation are taken from [12].

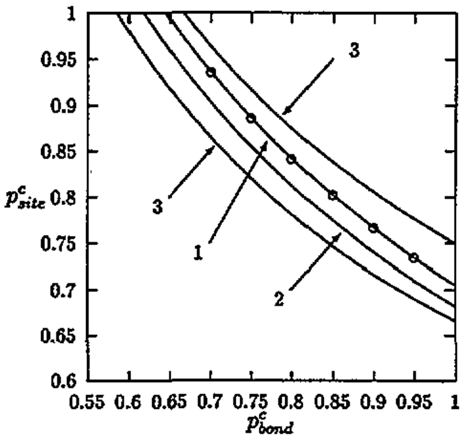

Figure 3. Critical line in 1+1 dimensions. Monte Carlo results are shown as points. (i) The critical line given by (2). (ii) The lower bound by Katori and Tsukahara [9], given as a solution of $\alpha^3\beta^4 - \alpha\beta^2 + 2\alpha\beta = 1$ with $\alpha = p^c_{site}$ and $\beta = p^c_{bond}$. (iii) The upper and lower bounds by Liggett [10], given by $\alpha = \frac{2+\beta}{4\beta}$ and $\alpha = \frac{2}{\beta(4-\beta)}$, respectively.

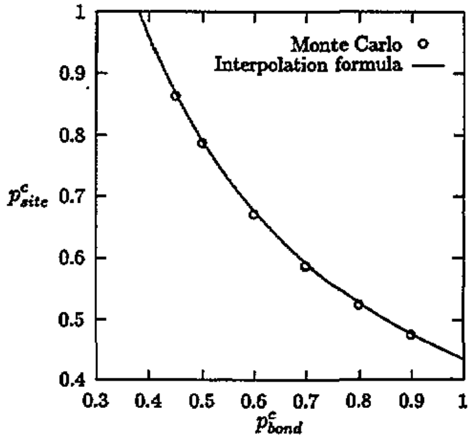

Figure 4. Critical line in 2+1 dimensions, for a simple cubic lattice.

directed percolation, we have

$$\frac{\log p^c_{site}}{\log p^{c^*}_{site}} + \frac{\log p^c_{bond}}{\log p^{c^*}_{bond}} = 1 \tag{2}$$

where $\log p^{c^*}_{site}$ and $\log p^{c^*}_{bond}$ are the critical fractions of bonds and sites in pure bond and site directed percolation.

We compare the critical line given by (2) with our Monte Carlo results in figures 3 and 4. The formula works surprisingly well, although if the difference between the interpolation formula prediction and the simulation results is plotted separately, as is done in figure 5, one can see that it cannot be explained by the flaws of the Monte Carlo simulation. Error bars, given in figure 5, are calculated as a combination of the uncertainty of the interpolation formula prediction due to the uncertainty of the threshold values for pure DP cases and of the error in p_{bond} values, obtained in our simulations. The critical values for pure DP cases were taken as $p^{c^*}_{site} = 0.705\,489 \pm 0.000\,004$ [13] and $p^{c^*}_{bond} = 0.644\,701 \pm 0.000\,001$ [15] in $1+1$ dimensions, and for $2+1$ dimensions we used the values obtained by [12],

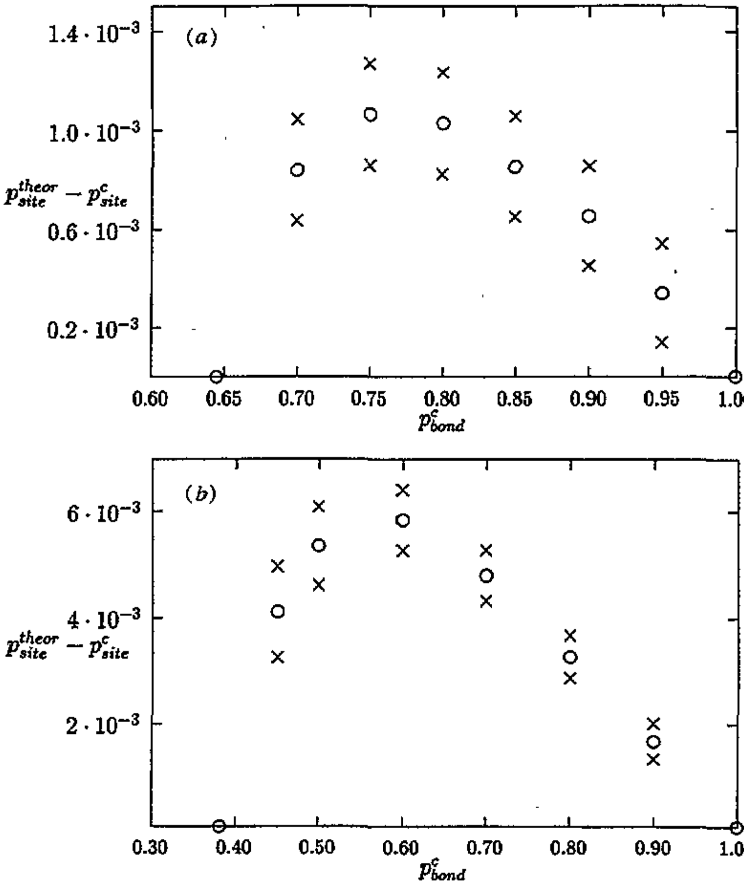

Figure 5. The difference between the predictions of (2) and the Monte Carlo results. Crosses indicate the uncertainty. (*a*) 1 + 1 dimensions. (*b*) 2 + 1 dimensions, simple cubic lattice.

$p_{site}^{c^*} = 0.435\,25 \pm 0.000\,13$ and $p_{bond}^{c^*} = 0.382\,16 \pm 0.000\,06$.

Apart from the interpolation formula, in figure 3 we compare the Monte Carlo results with exact boundaries for 1+1 mixed directed percolation, proposed by Katori and Tsukahara and by Liggett. Clearly, our results are in agreement with the bounds, presumed to be exact.

Concerning the universality of mixed site-bond directed percolation, both in 1 + 1 dimensions and in 2+1 dimensions, the dynamical exponents δ, η and z, obtained in our time-dependent simulations, were in agreement with the known DP values. For 1+1 dimensions, taking the DP exponents values given in [7], the error margins, estimated by analysing the spread of the local slopes data, were given as $\delta = 0.162 \pm 0.005$, $\eta = 0.308 \pm 0.005$ and $z = 1.263 \pm 0.010$. For 2 + 1 dimensions, using the DP exponents values from [12], the error margins were given as $\delta = 0.460 \pm 0.020$, $\eta = 0.214 \pm 0.025$ and $z = 1.134 \pm 0.010$. Thus, our results indicate that mixed directed percolation in 1 + 1 dimensions and in 2 + 1 dimensions belongs to the corresponding DP universality classes.

Concluding the results of our work, we would like to note that the interpolation formula suggested by Yanuka and Englman [2] for the ordinary mixed site-bond percolation works surprisingly accurately in the case of the directed mixed site-bond percolation as well. Nonetheless, the accuracy of our Monte Carlo results allows us to show a systematic difference. Our results in $1+1$ dimensions are in agreement with exact bounds suggested by Katori and Tsukahara [9] and by Liggett [10]. If the interpolation formula could be proven to be an upper boundary for the critical line as our results suggest, it would be a much closer bound than the Liggett's bound, the best available so far.

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